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MAGNETOELASTIC INSTABILITY OF STRUCTURES CARRYING ELECTRIC CURRENT†

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Abstract—The instability of a current carrying structure depends on the incremental inductance due to deformation. Numerical methods have been employed to compute the incremental inductances of simple elastic systems. The frequency current dispersion relations have been obtained for current-carrying beams and rings.

INTRODUCTION

The instability of a current-carrying elastic structure due to its own magnetic field is a new area in the field of mechanics. Leontovich and Shafranov[1] initiated the area by demonstrating that a flexible rod carrying electric current is inherently unstable for flexural motions. Later Dolbin[2] and Dolbin and Morozov[3] studied the flexural vibrations of an elastic rod and concluded that there exists a critical value of current exceeding which the rod cannot admit any vibratory motion. In all these papers, the authors assume that the current is confined on the surface of the rod. Chattopadhyay and Moon[4] have studied the vibration and stability of a current-carrying rod which has a uniform current distribution in its undeformed state. The existence of a critical current was established both analytically and experimentally in [4]. However, depending on the configuration of the current-carrying structure and its mode of vibration, the self magnetic field could be stabilizing or destabilizing.

In [1–4] the perturbed magnetic forces due to deformation or motion have been obtained in closed form. This is because of the simplicity of the body configuration. For complex structures carrying current it is not always easy to calculate the forces that result when the structure is deformed. For such structures, it is helpful to use an indirect procedure, namely the variational method. For non-dissipative current carrying systems (such as superconductors), it is possible to form a Lagrangian incorporating the kinetic energy, the potential energy and the magnetic energy. The stationary requirement of the Lagrangian yields the equations of motion.

In this work a variational method has been proposed for studying vibration and stability of current-carrying structures. The method has been applied to a simple structure like a simply-supported beam and the solution has been compared with the existing solutions for current-carrying beam. The method has also been applied to the problem of flexural vibrations of current-carrying rings. These results are applicable to the stability of small deformations of superconducting magnet coils for magnetic fusion reactors. A few analytical work and some experimental work in this general area have been reported by Moon *et al.*[5–8]. A comprehensive treatise in this area has been provided by Moon[9].

LAGRANGE'S EQUATIONS FOR CURRENT-CARRYING STRUCTURES

We consider a structure composed of conductors carrying electric currents. In the conductors the energy is stored as magnetic energy. The magnetic energy can be written as

$$W = 1/2 \sum \sum L_{ij} I_i I_j \tag{1}$$

where I_i, I_j are the currents through the ith and jth conductors respectively. L_{ij} is known as the inductance matrix, and it relates the magnetic energy and the current through a quadratic form.

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The kinetic energy T and the potential energy (or strain energy) can be formally written as

$$T = 1/2 \sum \sum M_{ij}\dot{X}_i\dot{X}_j \tag{2}$$

$$V = 1/2 \sum \sum K_{ij}X_iX_j \tag{3}$$

where M_{ij} and K_{ij} are the mass and stiffness matrices of the structure. X_i's are the amplitudes of the various modes of deformation that the structure is subjected to. If the generalized displacement of the structure is denoted by u, then

$$u = \dot{X}_i(t)\Phi_i(\xi) \tag{4}$$

where Φ_i are the mode shape functions.

We can form the Lagrangian assuming the currents are held constant, as

$$\mathscr{L}(X_i, \dot{X}_i) = T + W - V. \tag{5}$$

When no external forces are present, the equations of motion are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{X}_i}\right) - \frac{\partial \mathscr{L}}{\partial X_i} = 0. \tag{6}$$

Substituting the values of W, T and V from eqns (1)–(3) in eqns (5) and (6), we have

$$M_{ij}\ddot{X}_j + K_{ij}X_j - \frac{\partial W}{\partial X_i} = 0. \tag{7}$$

At this point we note that the inductance matrix is a function of motion or deformation. We can expand the inductance matrix about the equilibrium configuration (denoted by subscript o) as

$$L = L_o + \left.\frac{\partial L_{mn}}{\partial X_k}\right|_o X_k + \left.\frac{\partial^2 L_{mn}}{\partial X_i \partial X_j}\right|_o X_i X_j \tag{8}$$

where we have neglected the higher order terms.

The equation of motion (7), using the eqns (1) and (8) now takes the following form:

$$M_{ij}\ddot{X}_j + \left(K_{ij} - \left.\frac{\partial^2 L_{mn}}{\partial X_i \partial X_j}\right|_o I_m I_n\right) X_j = \left.\frac{\partial L_{mn}}{\partial X_i}\right|_o I_m I_n. \tag{9}$$

The right hand side of eqn (9) represents an initial magnetic force on the system. If the systems are such that the magnetic forces balance each other in the undeformed configuration, then there is no initial magnetic force. For such assumed systems we can set

$$\left.\frac{\partial L_{mn}}{\partial X_i}\right|_o = 0. \tag{10}$$

FREQUENCY-CURRENT DISPERSION

In order to study the frequency-current dispersion relation we look for amplitudes harmonically varying with time, thus

$$X_j = \bar{X}_j \, \mathrm{e}^{i(\omega t + \Phi)}. \tag{11}$$

Substituting X in eqn (9) and using eqn (10), we observe that for non-trivial solutions for X_j's the following determinant must vanish.

$$\left| k_{ij} - \frac{\partial^2 L_{mn}}{\partial X_i \partial X_j}\bigg|_o I_m I_n - \omega^2 M_{ij} \right| = 0. \tag{12}$$

The eqn (12) is the frequency-current dispersion equation. When the currents are zero, this equation gives the conventional structural frequencies. When $\omega \to 0$, the structure fails to admit any vibratory motion and "buckles". This happens when

$$\left| K_{ij} - \frac{\partial^2 L_{mn}}{\partial X_i \partial X_j}\bigg|_o I_m I_n \right| = 0. \tag{13}$$

For a single vibratory mode, the eqn (12) simplifies to

$$K_{11} - \frac{\partial^2 L_{11}}{\partial X_1^2}\bigg|_o I_1^2 - \omega^2 M_{11} = 0 \tag{14}$$

or,

$$\omega^2 = \omega_o^2 - (l_1/m) I^2 \tag{15}$$

where,

$$\omega_0 = \frac{K_{11}}{M_{11}}, \quad \text{the zero current natural frequency}$$

$$m \equiv M_{11}$$

$$I \equiv I_1$$

$$l_1 \equiv \frac{\partial^2 L_{11}}{\partial X_1^2}.$$

Here l_1 refers to the incremental inductance coefficient due to motion and is given by

$$L = L_o + l_1 A^2 \tag{16}$$

where A is the amplitude of vibration. This result follows from eqns (8) and (10).

From eqn (15) we note that if l_1 is positive, the frequency of vibration decreases with increasing current, thus indicating an unstable situation. For negative values of l_1 the structure stiffens with increasing current as evidenced by the increase in natural frequency.

STABILITY OF A CURRENT-CARRYING ROD

The previous studies on the stability of vibration of a current-carrying rod is considered the perturbed magnetic fields due to deformation and obtained the destabilizing magnetic forces [1–3]. In this section the incremental inductance approach is used to obtain the frequency-current dispersion. In order to compute the incremental inductance we take a rod of finite length and subject it to a sinusoidal deformation. The wavelength of deformation is taken as twice the length of the rod. The self inductance of the rod is computed numerically for the deformed and the undeformed configurations. The contributions to the self inductance of the rod comes from the regions inside and outside the rod. The internal part of the inductance is dependent on the geometry of the cross section of the rod, and is assumed not to change with the deformation. The incremental inductance is then the difference in the external part of the inductances for the deformed and the undeformed configurations.

To get the external part of the inductance, we need to find the mutual inductance of two curvilinear circuits spaced at a distance equal to the radius of the rod (see, e.g. Ref. [16]).

$$L = \frac{\mu_o}{4\pi} \int \int \frac{\mathrm{d}s_1 \cdot \mathrm{d}s_2}{D} \tag{17}$$

where ds_1 is the differential element of the filament running along the rod axis, ds_2 is a differential element along its inner edge. D is the distance between the filaments. μ_o is the permeability.

The deformation of the rod is specified in the following manner:

$$y(x, t) = \delta(t) \sin(\pi x/l) \tag{18}$$

where y is the transverse displacement, l is the length of the rod having the ends at $x = 0$ and $x = l$. $\delta(t)$ measures the amplitude of deformation corresponding to the sinusoidal mode shape.

The inductances are evaluated using eqn (17) for undeformed and deformed configurations. For deformed configurations various values of δ are used in the numerical computations. The details of the numerical procedures can be found in [10]. The results are shown in Fig. 1.

We have, the inductance for a particular δ as

$$L = L_o + l_1\delta^2 \tag{19}$$

where L_o is the inductance corresponding to the undeformed configuration and l_1 is the incremental inductance coefficient.

From Fig. 1 the incremental inductance coefficient l_1 is obtained as 4×10^{-6} H per m^2. The positive value of l_1 indicates that a critical current exists. From eqn (15) we have

$$(\omega_o^2 - \omega^2) = l_1 I^2/m. \tag{20}$$

Here m is the modal mass of the rod and equals to half its total mass by virtue of the defined mode shape in eqn (18). From eqn (20) buckling occurs when $\omega = 0$, which gives the critical current I_* as

$$I_*^2 = \frac{\omega_o^2 m}{l_1} = \frac{EJ\pi^4}{2l^3 l_1} \tag{21}$$

where we have substituted $\omega_o^2 = (EJ\pi^4)/(\rho A l^4)$, the zero current natural frequency for the first mode of flexural vibration and $m = 1/2\rho A l$, the modal mass of the rod. EJ measures the flexural rigidity of the rod, ρ, l and A are the mass density, the length and the cross-sectional area of the rod respectively.

Chattopadhyay and Moon[4] obtain the critical current of a rod carrying uniformly dis-

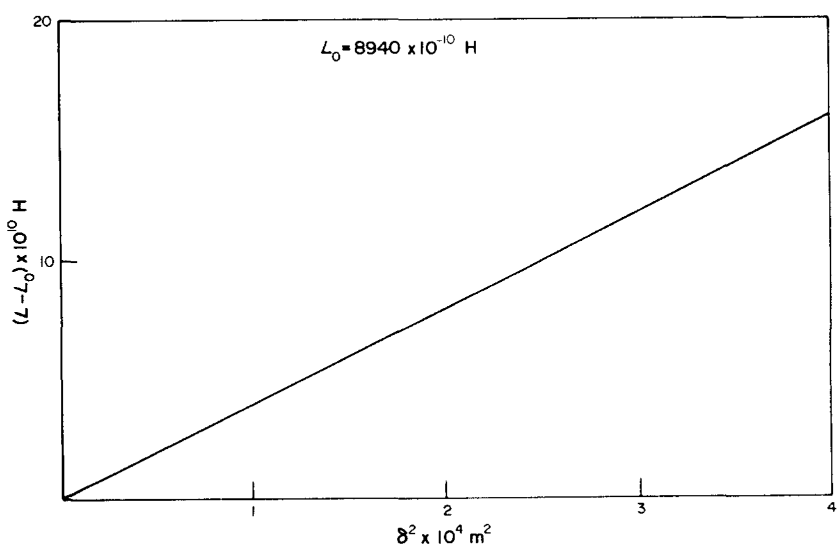

Fig. 1. Incremental inductance of a rod.

tributed current as

$$I_*^2 = \frac{EJ(\pi/l)^2}{\mu_o/4\pi[\ln(2l/\rho a) - 1.327]} \tag{22}$$

where a is the radius of the rod.

For the case of axial surface current, the critical current obtained by Dolbin and Morozov is identical except for the additive numerical constant in the denominator of eqn (22). Their numerical constant is 1.077 as opposed to 1.327 in eqn (22)[3].

Numerical values

$$\begin{aligned} l &= 0.762\ \text{m} \\ a &= 0.00159\ \text{m} \\ E &= 8.7 \times 10^{10}\ \text{N/m}^2 \\ \rho &= 8400\ \text{kg/m}^3 \\ J &= (\pi a^4/4) = 5 \times 10^{-12}\ \text{m}^4 \\ \mu_o &= 4\pi \times 10^{-7}\ \text{H/m.} \end{aligned}$$

With the above numerical values, the critical currents are calculated as

$I_* = 3460$ amp (numerical method, eqn (21))
$I_* = 4100$ amp (Chattopadhyay and Moon [4])
$I_* = 3990$ amp (Dolbin and Morozov [3]).

Hence the numerical results are conservative as far as the critical currents are concerned.

STABILITY OF CIRCULAR RINGS CARRYING ELECTRIC CURRENT

The shape of the superconducting coils for magnets in fusion reactors might be in the form of circular rings. A number of such coils are assembled around the periphery of a cylindrical structure to form the magnet. These coils carry very high currents to produce a toroidal field for plasma confinement. It is very important to investigate the forces resulting from the motion or deformation of such coils to see whether such forces can have a destabilizing effect on the coils. The study will be confined to the elastic stability of a single coil carrying current due to the self-magnetic field. To study the stability of motion of such coils, it is necessary to know the inductance increment of the coils under deformation. In this section we study the deformation in the ring plane of the ring as well as the out-of-plane deformation of the ring. For these deformations, the incremental inductances are evaluated numerically.

The inductance of the ring has an internal part and an external part. The internal part of the inductance, a function of the cross section is assumed not to change with deformation. The external part of the inductance is the mutual inductance of two circuits, one running along the central axis of the ring and the other along the inner edge [11].

The in-plane deformation is specified as

$$u_r = F \cos 2\theta \tag{23}$$

where F is the amplitude of the radial (in-plane) deformation and θ is circumferential coordinate.

For different values of F the values of inductances are computed numerically. The mathematical expression of the inductance L is given by

$$L = \frac{\mu_o}{4\pi} \int \int \frac{\mathbf{ds}_a \cdot \mathbf{ds}_i}{|\mathbf{r}_a - \mathbf{r}_i|} \tag{24}$$

where $\mathbf{ds}_a$ is a differential distance along the central axis of the ring having a position vector $\mathbf{r}_a$, and $\mathbf{ds}_i$ is a differential distance along the inner edge of the coil having a position vector $\mathbf{r}_i$.

The results of the numerical computation corresponding to the in-plane motion of the ring are shown in Fig. 2. We have

$$L = L_o + l_{ip}F^2 \tag{25}$$

where L_o is the inductance of the ring in its undeformed configuration and l_{ip} the incremental inductance coefficient for the in-plane deformation.

The out-of-plane deformation is specified as

$$V = C\cos 2\theta \tag{26}$$

where v is the displacement normal to the plane of the ring and C is the amplitude of deformation.

For different values of C, the inductances are computed numerically using eqn (24). The results are shown in Fig. 3. We have

$$L = L_o + l_{op}C^2 \tag{27}$$

where l_{op} is the incremental inductance coefficient for the motion normal to the plane of the ring.

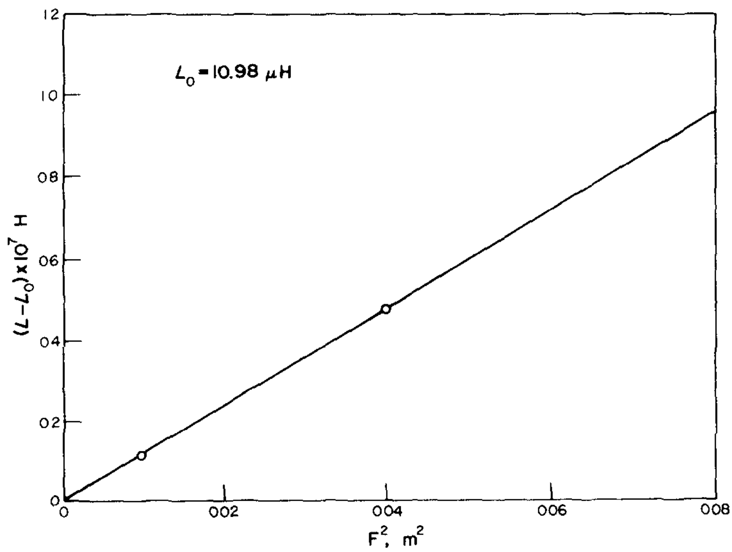

Fig. 2. Incremental inductance for in-plane vibration of a ring.

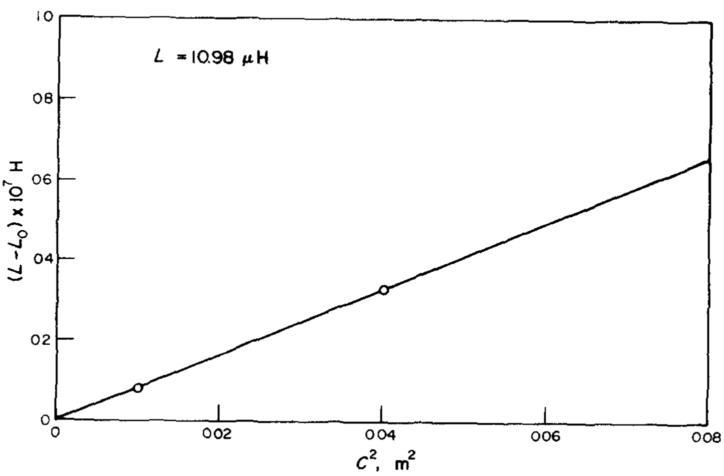

Fig. 3. Incremental inductance for out-of-plane vibration of a ring.

NUMERICAL VALUES

As an example, we take the design data of a fusion reactor coil as given in Ref. [12]. We approximate the coil shape by a ring of radius R_o with a cross-sectional radius r. The inner edge of the coil is a circle of radius $R_o - r$, where, $R_o = 3.03$ m and $r = 0.165$ m. We then get the following numerical results

$$l_{ip} = 1.19 \times 10^{-6}\ \mathrm{H/m^2}\ \text{(Fig. 2)}$$
$$l_{op} = 0.83 \times 10^{-6}\ \mathrm{H/m^2}\ \text{(Fig. 3)}.$$

Evaluation of the initial magnetic tension

A current-carrying ring in its undeformed configuration is subjected to a radial magnetic force distribution which gives rise to a hoop tension in the ring. An approximate value of this circumferential tension T_θ has been calculated in Ref. [13] and is

$$T_\theta = \frac{\mu_o I^2}{4\pi}[\ln(8R_o/r) - 3/4]. \tag{28}$$

The magnitude of this tension can be estimated numerically from the magnetic energy $W = 1/2LI^2$ as

$$T_\theta = \frac{1}{2\pi}\frac{\partial W}{\partial R_o} = \frac{I^2}{4\pi}\frac{\partial L}{\partial R_o}. \tag{29}$$

The derivative $\partial L/\partial R_o$ is numerically computed at $R_o = 3.03$ m by evaluating inductances for R_o values slightly less and slightly greater than 3.03 m. The inductance is graphically represented as a function of R_o in Fig. 4. The derivative is given by the slope of the straight line curve in Fig. 4. Thus the partial derivative $\partial L/\partial R_o$, and hence the initial magnetic tension is known from eqn (29).

FREQUENCY-CURRENT DISPERSION FOR IN-PLANE VIBRATION OF A CIRCULAR RING

The deformation has been specified for the radial displacement u_r in eqn (23). If we are interested in flexural vibration without extension, then the circumferential displacement u_θ must be related to the radial displacement by [14]

$$\frac{\partial u_\theta}{\partial \theta} = u_r \tag{30}$$

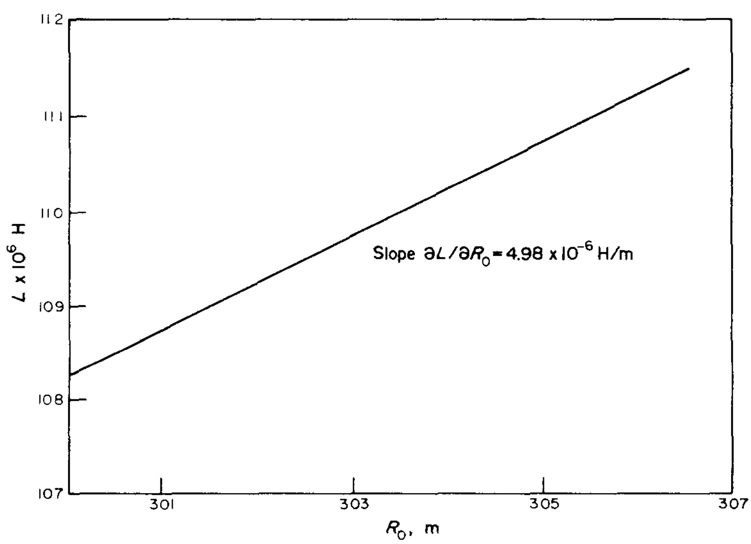

Fig 4 Inductance vs radius of the ring

From eqn (23), u_θ is therefore given by

$$u_\theta = \frac{F}{2}\sin 2\theta. \tag{31}$$

The strain energy of the ring, V, including the effect of initial tension T_θ is given by

$$V = \frac{EJ}{2R_o}\int_o^{2\pi}\left(\frac{\partial^2 u_r}{\partial\theta^2} + u_r\right)^2 R_o\,\mathrm{d}\theta + \frac{T_\theta}{2R_o^2}\int_o^{2\pi}\left(\frac{\partial u_r}{\partial\theta}\right)^2 R_o\,\mathrm{d}\theta$$

$$= \frac{9EJ\pi}{2R_o^3}F^2 + \frac{2T_\theta\pi}{R_o}F^2. \tag{32}$$

The kinetic energy, T, of the ring is

$$T = \frac{\rho A}{2}\int_o^{2\pi}(\dot{u}_r^2 + \dot{u}_\theta^2)R_o\,\mathrm{d}\theta = \frac{5\pi}{8}\rho A R_o \dot{F}^2. \tag{33}$$

The magnetic energy of the ring using eqn (25) is

$$W = 1/2(L_o + l_{ip}F^2)I^2. \tag{34}$$

With the Lagrangian $\mathscr{L} = T + W - V$, the equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{F}}\right) - \frac{\partial\mathscr{L}}{\partial F} = 0 \tag{35}$$

which then gives, using eqns (32)–(34) and (29)

$$\ddot{F} + \frac{36EJ\pi}{5\rho A R_o^4}\left[1 + \frac{R^3 I^2}{9\pi EJ}\left(\frac{l}{R_o}\frac{\partial L}{\partial R_o} - l_{ip}\right)\right]F = 0. \tag{36}$$

Using the notation

$$l_{\mathrm{eff}} = \frac{l}{R_o}\frac{\partial L}{\partial R_o} - l_{ip}. \tag{37}$$

From eqn (36), the frequency-current dispersion is obtained as

$$\omega^2 = \omega_o^2\left(1 + \frac{l_{\mathrm{eff}}R_o^3 I^2}{9EJ\pi}\right) \tag{38}$$

where

$$\omega_o = \left(\frac{36EJ}{5\rho A R_o^4}\right)^{1/2} \tag{39}$$

the zero current natural frequency.

From eqn (38), if l_{eff} is positive, the natural frequency of the ring increases with the current and a stable situation results. If the initial tension effect is neglected then $l_{\mathrm{eff}} = -l_{ip}$ from eqn (37). Then for positive values of l_{ip} the ring natural frequency decreases with increasing current thus indicating an unstable situation. Then a critical current exists exceeding which the ring cannot admit any vibratory motion and buckles at sufficiently high current. The in-plane buckling current corresponding to the case of no initial magnetic tension is given by

$$I_*^2 = \frac{9EJ\pi}{l_{ip}R_o^3}. \tag{40}$$

The frequency current dispersion relation as given by eqn (38) now takes the following form using eqn (40)

$$\omega^2 = \omega_o^2 \left(1 + \frac{l_{\text{eff}} I^2}{l_{tp} I_*^2}\right) \tag{41}$$

with l_{eff} and I_* given by eqns (37) and (40) respectively.

NUMERICAL VALUES FOR IN-PLANE VIBRATION OF A CURRENT-CARRYING RING

$$\begin{aligned}
E &= 8 \times 10^{10}\,\text{N/m}^2 \\
J &= 2.2 \times 10^{-4}\,\text{m}^4 \\
R_o &= 3.03\,\text{m} \\
l_{tp} &= 1.19 \times 10^{-6}\,\text{H/m}^2\ \text{(Fig. 2)} \\
I_* &= 3.877 \times 10^6\,\text{amp (from eqn 40)} \\
\frac{\partial L}{\partial R_o} &= 4.98 \times 10^{-6}\,\text{H/m (Fig. 4)} \\
l_{\text{eff}} &= 0.45 \times 10^{-6}\,\text{H/m}^2\ \text{(from eqn 37).}
\end{aligned}$$

Using the above numerical values, the frequency current dispersion relation (41) now reads

$$\omega^2 = \omega_o^2(1 + 2.51 \times 10^{-14} I^2). \tag{42}$$

FREQUENCY-CURRENT DISPERSION FOR OUT OF PLANE VIBRATION OF A RING

The transverse displacement v has been specified in eqn (26). For out of plane motion and twist ϕ is coupled with v. For a ring undergoing combined bending and torsional motion, the strain energy, V, the kinetic energy T, and the magnetic energy W are given by

$$V = \frac{EJ}{2}\int_0^{2\pi}\left(\frac{1}{R_o} + \frac{1}{R_o^2}\frac{\partial^2 v}{\partial\theta^2}\right)^2 R_o\,\mathrm{d}\theta + \frac{GK_t}{2}\int_0^{2\pi}\left(\frac{1}{R_o}\frac{\partial\phi}{\partial\theta} + \frac{1}{R_o^2}\frac{\partial^2 v}{\partial\theta^2}\right)^2 R_o\,\mathrm{d}\theta + \frac{T_\theta R_o}{2}\int_0^{2\pi}\left(\frac{1}{R_o}\frac{\partial v}{\partial\theta}\right)^2 \mathrm{d}\theta \tag{43}$$

$$T = 1/2\rho A \int_0^{2\pi} \dot{v}^2 R_o\,\mathrm{d}\theta \tag{44}$$

$$W = 1/2(L_o + l_{op}C^2)I^2 \tag{45}$$

where G is the shear modulus and K_t is the torsion constant and for a circular section equals $2J$. In eqn (43) we have included the contribution due to the initial magnetic tension whose value is given by eqn (29). In eqn (44) we have neglected the rotatory inertia. In eqn (45) we have implicitly assumed that the twist ϕ has no effect on the inductance, and hence the magnetic energy.

For the twist we assume an identical mode shape as for the displacement given in eqn (26). Thus

$$\phi = D\cos 2\theta \tag{46}$$

where D is the amplitude of the twist.

Substituting v and ϕ from eqns (26) and (46) in eqns (43) and (44), we have

$$V = \frac{\pi EJR_o}{2}\left(\frac{D^2}{R_o^2} + \frac{8CD}{R_o^3} + \frac{16C^2}{R_o^4}\right) + \frac{\pi GK_t R_o}{2}\left(\frac{4D^2}{R_o^2} + \frac{8CD}{R_o^3} + \frac{4C^2}{R_o^4}\right) + \frac{\pi T_\theta R_o}{2}\frac{4C^2}{R_o^2} \tag{47}$$

$$T = \frac{\rho A \pi R_o \omega^2}{2} C^2. \tag{48}$$

The Lagrangian $\mathscr{L}$ is given by $T+W-V$. For $\mathscr{L}$ to be stationary, we must have

$$\frac{\partial \mathscr{L}}{\partial C}=\frac{\partial \mathscr{L}}{\partial D}=0. \tag{49}$$

Equation (49) gives us a set of two homogeneous simultaneous equations in C and D. For non-trivial solutions, the determinant of the coefficient matrix should vanish. This gives

$$\begin{vmatrix} l_{\text{eff}}I^2-\rho A\omega^2 R_o+\frac{4}{R_o^3}EJ(4+\beta) & \frac{4EJ}{R_o^2}(1+\beta) \\ \frac{4EJ}{R_o^2}(1+\beta) & \frac{EJ}{R_o^2}(1+4\beta) \end{vmatrix}=0 \tag{50}$$

where,

$$l_{\text{eff}}=\frac{1}{R_o}\frac{\partial L}{\partial R_o}-l_{op} \tag{51}$$

and,

$$\beta=\frac{GK_t}{EJ}=\frac{1}{1+\nu} \tag{52}$$

where ν is the Poisson's ratio.

Equation (50) yields the following frequency-current dispersion relation,

$$\omega^2=\omega_o^2\left[1+\frac{R_o^3(5+\nu)}{36EJ}l_{\text{eff}}I^2\right] \tag{53}$$

where,

$$\omega_o=\left(\frac{36EJ\pi}{(5+\nu)AR_o^4}\right)^{1/2} \tag{54}$$

ω_o is the zero current natural frequency for the flexural vibration at right angle to the plane of the ring. This value checks with the corresponding result obtained in Ref. [15].

From eqn (53), if l_{eff} is positive, the out-of-plane natural frequency increases with increasing current. If the initial magnetic tension is neglected, then $l_{\text{eff}}=-l_{op}$ from eqn (51). Then for positive values of l_{op} a critical current exists. The out-of-plane buckling current for the case of zero initial tension is given by

$$I_*^2=\frac{36EJ\pi}{l_{op}(5+\nu)R_o^3}. \tag{55}$$

The frequency-current dispersion relation now takes the following form

$$\frac{\omega^2}{\omega_o^2}=1+\frac{l_{\text{eff}}}{l_{op}}\frac{I^2}{I_*^2}. \tag{56}$$

NUMERICAL RESULTS FOR THE OUT-OF-PLANE VIBRATIONS OF A CURRENT-CARRYING RING

$E=8\times10^{10}\ \text{N/m}^2$
$J=2.2\times10^{-4}\ \text{m}^4$
$R_o=3.03\ \text{m}$
$l_{op}=0.83\times10^{-6}\ \text{H/m}^2$ (Fig. 3)

$$I_* = 4.1 \times 10^6 \text{ amp (from eqn 55)}$$

$$\frac{\partial L}{\partial R_o} = 4.98 \times 10^{-6} \text{ H/m (Fig. 4)}$$

$$l_{\text{eff}} = 0.81 \times 10^{-6} \text{ H/m}^2 \text{ (from eqn 51)}.$$

Using the numerical values the frequency-current dispersion relation (56) now reads

$$\frac{\omega^2}{\omega_0^2} = 1 + 6.0 \times 10^{-14} I^2. \tag{57}$$

Thus the destabilizing effect due to the self field tends to be exceeded by the increased initial magnetic tension in the ring. An experimental result confirming this effect appears in Ref. [8].

The stabilizing effect of the initial magnetic tension rests on the assumption that the center of tension coincides with the center of current. In some fusion reactor coil geometries this may not hold. The non-coincidence of the two centers can cause additional forces and moments requiring further analyses.

Further, in a magnetic fusion reactor the motion of a coil is influenced by the fields in the neighboring coils. The destabilizing effects due to the fields external to the coil is reported in Refs. [6–8].

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